Hausdorff Continuous Interval Functions and Approximations

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Abstract. The set of interval Hausdorff continuous functions constitutes the largest space preserving basic algebraic and topological structural properties of continuous functions, such as linearity, ring structure, Dedekind order completeness, etc. Spaces of interval functions have important applications not only in the construction of numerical methods and algorithms, but to problems in abstract areas such as real analysis, set-valued analysis, approximation theory and the analysis of PDEs. In this work, we summarize some basic results about the family of interval Hausdorff continuous functions that make interval analysis a bridge between numerical and real analysis. We focus on some approximation issues formulating a new result on the Hausdorff approximation of Hausdorff continuous functions by interval step functions. The Hausdorff approximation of the Heaviside interval step function by sigmoid functions arising from biological applications is also considered, and an estimate for the Hausdorff distance is obtained.

Keywords. Interval functions, Baire semi-continuous functions, Hausdorff continuous functions, Dilworth continuous functions, sigmoid functions.

1 Introduction

Functions having discontinuities are encountered in many situations. A widely used class of discontinuous functions is the class of Baire upper (lower) semi-continuous functions [13]. Dilworth restricted the class of Baire semi-continuous functions up to normal semi-continuous functions [15]. Both classes are conveniently reformulated in terms of interval-valued functions using graph completion operators [3]. For example, when a graph is completed, the Dilworth normal semi-continuous functions are the Hausdorff continuous interval-valued functions. Such a reformulation leads to interesting original results oriented to practical applications.

It has been shown that the space of Hausdorff continuous functions is the largest linear space of interval functions [3]–[5], [22]. This space has important applications in the theory of PDEs and real analysis [1], [6]–[9], [12]. Moreover, the space of Hausdorff continuous functions has a special place in *Interval Analysis* as well, more specifically in the analysis of interval-valued functions [2]. It has been also shown that the practically relevant set, in terms of providing tight enclosures of sets of real continuous functions, is the set of Dilworth continuous interval-valued functions [5].

The relation between Baire semi-continuous functions and interval-valued functions establishes a new paradigm for *interval analysis* as part of *analysis* rather than (or in addition to) the field of *numerical methods*, where it is currently classified, see e.g. [23]. This provides a new research direction of applying interval analysis to abstract mathematical problems. It has been shown that the spaces of interval-valued functions[3] have important applications not only in the construction of numerical methods but to problems in more abstract areas like real analysis, set-valued analysis, approximation theory and analysis of PDEs. Some of the more interesting results are: (i) A generalization of the order convergence structure on the space of Hausdorff continuous functions to a convergence structure on the space of minimal upper semi-continuous compact set-valued (shortly: usco) maps [10]; (ii) All rational extensions and their metric completions of $C(X)$ are subspaces of the space of Hausdorff continuous functions [11]; (iii) The solutions of large classes of nonlinear systems of PDEs can be presented with Hausdorff continuous interval functions [6], [7]; (iv) The theory of continuous viscosity solutions of Hamilton-Jacobi equations can be recast in the setting of Hausdorff continuous functions, where the discontinuous solutions are accommodated in a natural way [8].

In the next Section 2 we summarize some basic results concerning the class of interval Hausdorff continuous functions and the related classes of interval functions. Section 3 contains a new result on the Hausdoff approximation of Hausdorff continuous functions by interval step functions. Section 4 is devoted to a new result on the approximation of interval step functions by a class of sigmoid functions arising from biological applications.

2 Classes of interval functions: basic results

The concept of Hausdorff continuity (H-continuity) generalizes the familiar concept of continuity in such a way that many essential properties of the usual continuous real functions are preserved. The set $C(\Omega)$ of all continuous real functions defined on a subset $\Omega \subset \mathbb{R}^n$ is a commutative ring with respect to the point-wise defined addition and multiplication of functions and a linear space with respect to addition and multiplication by a scalar. Is it possible to extend the algebraic operations on $C(\Omega)$ to the set $\mathbb{H}(\Omega)$ of H-continuous functions in a way that preserves these two basic algebraic structures, that is, the set $\mathbb{H}(\Omega)$ to become a commutative ring and linear space with respect to the extended operations? It turns out that the answer is affirmative as briefly shown in the sequel.

2.1 Basic notation and definitions: Baire continuous functions

Definition 1. *[13] A real-valued function f is* upper (lower) semi-*continuous at a point x_0 if the function values for arguments near x_0 are* either *close to $f(x_0)$* or less (greater) than $f(x_0)$.

Intervals on the real line $\mathbb{R}$ are denoted as $a = [\underline{a}, \overline{a}] = \{x : \underline{a} \le x \le \overline{a}\}$, and the set of all intervals is denoted $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R},\ \underline{a} \le \overline{a}\}$; denote also $w(a) = \overline{a} - \underline{a}$, and $|a| = \max\{|\underline{a}|, |\overline{a}|\}$.

[3] for brevity we shall further write “interval function” instead of “interval-valued function”

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A real or interval function f on Ω is locally bounded if for every $x \in \Omega$ there exist $\delta > 0$ and $M \in \mathbb{R}$ such that $|f(y)| < M,\ y \in B_\delta(x),\ B_\delta(x) = \{y \in \Omega : ||x-y|| < \delta\}$. Denote

$$\mathbb{A}(\Omega) = \{f : \Omega \to \mathbb{I}\,\mathbb{R},\ f \text{ locally bounded}\},$$
$$\mathcal{A}(\Omega) = \{f : \Omega \to \mathbb{R},\ f \text{ locally bounded}\} \ \subseteq\ \mathbb{A}(\Omega).$$

Definition 2. *D is a dense subset of Ω. The lower/upper Baire operators $I(D,\Omega,\cdot)$, $S(D,\Omega,\cdot):\ \mathbb{A}(D) \to \mathcal{A}(\Omega)$ are defined for $f = [\underline{f},\ \overline{f}] \in \mathbb{A}(D)$ and $x \in \Omega$ by*

$$I(D,\Omega,f)(x) = \sup_{\delta>0} \inf\ \{\underline{f}(y) :\ y \in B_\delta(x) \cap D\},$$

$$S(D,\Omega,f)(x) = \inf_{\delta>0} \sup\ \{\overline{f}(y) :\ y \in B_\delta(x) \cap D\}.$$

Definition 3. *The graph completion operator $F : \mathbb{A}(D) \to \mathbb{A}(\Omega)$ for $f \in \mathbb{A}(D)$ is defined as*

$$F(D,\Omega,f)(x) = [I(D,\Omega,f)(x),\ \ S(D,\Omega,f)(x)],\ \ x \in \Omega,\ \ f \in \mathbb{A}(D).$$

For $D = \Omega$ we write

$$I(f) = I(\Omega,\Omega,f),\ \ S(f) = S(\Omega,\Omega,f),\ \ F(f) = F(\Omega,\Omega,f).$$

Using end-point presentation of functions: $f = [\underline{f},\ \overline{f}] \in \mathbb{A}(\Omega)$ we can write

$$I(D,\Omega,f) = I(D,\Omega,\underline{f}), \quad S(D,\Omega,f) = S(D,\Omega,\overline{f}),$$
$$F(D,\Omega,f) = [I(D,\Omega,\underline{f}),\ S(D,\Omega,\overline{f})].$$

Definition 4. *A function $f \in \mathbb{A}(\Omega)$ is* S-continuous*, if $F(f) = f$.*

Definition 5. *A function $f \in \mathbb{A}(\Omega)$ is* D-continuous*, if for every dense subset D of Ω, $F(D,\ \Omega,\ f) = f$.*

Definition 6. *A function $f \in \mathbb{A}(\Omega)$ is* H-continuous*, if for every function $g \in \mathbb{A}(\Omega)$ such that $g(x) \subseteq f(x),\ \ x \in \Omega$, $F(g)(x) = f(x),\ \ x \in \Omega$.*

Theorem 1. *For every $f \in \mathbb{H}(\Omega)$ the set $W_f = \{x \in \Omega :\ w(f(x)) > 0\}$ is of first Baire category (that is, H-continuous functions are "thin").*

H-continuous functions do not differ much from the usual real-valued continuous functions because they assume interval values only on a meagre set[4].

2.2 Arithmetic operations in $\mathbb{H}(\mathbb{R})$

Interval arithmetic operations are denoted as usually: for $a = [\underline{a},\overline{a}], b = [\underline{b},\overline{b}] \in \mathbb{I}\,\mathbb{R}$ we have $a+b = \{\alpha+\beta : \alpha \in a,\ \ \beta \in b\}$, $a \times b = \{\alpha\beta : \alpha \in a,\ \ \beta \in b\}$; or endpoint-wise: $[\underline{a},\overline{a}]+[\underline{b},\overline{b}] = [\underline{a}+\underline{b},\ \overline{a}+\overline{b}]$, $[\underline{a},\overline{a}]\times[\underline{b},\overline{b}] = [\min\{\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}\},\ \max\{\underline{a}\underline{b},\underline{a}\overline{b},\overline{a}\underline{b},\overline{a}\overline{b}\}]$.

[4] In topology, a *meagre set*, also called a *set of first Baire category*, is a set that, considered as a subset of a (usually larger) topological space, is *small or negligible.*

For functions $f, g \in \mathbb{A}(\Omega),\ f = [\underline{f}, \overline{f}],\ \ g = [\underline{g}, \overline{g}],\ \ x \in \Omega$, we have [2]

$$\begin{aligned}(f+g)(x) = f(x) + g(x)\ &= \ [\underline{f}(x) + \underline{g}(x), \overline{f}(x) + \overline{f}(x)],\\ (f \times g)(x) = f(x) \times g(x) &= [\min M, \max M],\\ M = \{\underline{f}(x)\underline{g}(x), &\underline{f}(x)\overline{g}(x), \overline{f}(x)\underline{g}(x), \overline{f}(x)\overline{g}(x)\}.\end{aligned}$$

Example 1. Denote by $h \in \mathbb{H}(\mathbb{R})$ the (interval) Heaviside step function given by

$$h(x) = \begin{cases} 0, & \text{if } x < 0,\\ [0,1], & \text{if } x = 0,\\ 1, & \text{if } x > 0, \end{cases} \tag{1}$$

and $g = (-1) \times h \in \mathbb{H}(\mathbb{R})$. For the sum $h + g$ we have

$$(h+g)(x) = h(x) + g(x) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x > 0\\ [-1,1], & \text{if } x = 0, \end{cases}$$

showing that $h + g \notin \mathbb{H}(\mathbb{R})$.

Theorem 2. *(a) There exists a unique function $p \in \mathbb{H}(\Omega)$ such that $p(x) \subseteq (f + g)(x),\ \ x \in \Omega$; (b) There exists a unique function $q \in \mathbb{H}(\Omega)$ such that $q(x) \subseteq (f \times g)(x),\ \ x \in \Omega$.*

We define H-addition and H-multiplication of H-continuous functions $f, g \in \mathbb{H}(\Omega)$ via interval operations as follows.

Definition 7. *(a) $f \oplus g$ is the unique H-continuous function $p(x)$ as defined by Theorem 2 (a), that is satisfying $(f \oplus g)(x) \subseteq (f + g)(x),\ \ x \in \Omega$; (b) $f \otimes g$ is the unique H-continuous function $q(x)$ as defined by Theorem 2 (b), that is satisfying $(f \otimes g)(x) \subseteq (f \times g)(x),\ \ x \in \Omega$.*

Example 2. For the H-sum of the Heaviside step function $h \in \mathbb{H}(\mathbb{R})$ given by (1) and $g = (-1) \times h \in \mathbb{H}(\mathbb{R})$ we have $(h \oplus g)(x) = 0,\ \ x \in \mathbb{R}$.

Theorem 3. *The set $\mathbb{H}(\Omega)$ is a commutative ring with identity with respect to the H-operations $\oplus$ and $\otimes$.*

Remark. Note that the H-operations $\oplus$ and $\otimes$ are not point-wise in general. At a point where both operands have interval values the value of the H-sum $\oplus$ or the H-product $\otimes$ are not determined by the values of the operands only at that point but rather by the values of the operands in a neighborhood of the point.

In the special case when one of the operands is a real (point) valued function the operations $\oplus$ and $\otimes$ coincide with the point-wise operations, namely we have:

$$\begin{aligned}(f \oplus g)(x) &= (f + g)(x) \ \text{ if } \ w(f(x)) = 0 \text{ or } w(g(x)) = 0,\\ (f \otimes g)(x) &= (f \times g)(x) \ \text{ if } \ w(f(x)) = 0 \text{ or } w(g(x)) = 0.\end{aligned}$$

More properties of the H-operations can be found in [4].

2.3 The set of H-continuous functions as a linear space

Multiplication by a scalar is defined as multiplication by a constant function. Since the value of this function is a real number this multiplication coincides with the point-wise multiplication

$$(\alpha * f)(x) = \alpha * f(x) = \begin{cases} [\alpha\underline{f}(x), \alpha\overline{f}(x)] \text{ if } \alpha \geq 0 \ , \\ [\alpha\overline{f}(x), \alpha\underline{f}(x)] \text{ if } \alpha < 0. \end{cases}$$

The set $\mathbb{H}(\Omega)$ is a linear space with respect to "$\oplus$" and "$*$". Moreover, it is the largest space of interval functions as stated in the next theorem.

Theorem 4. *[5] Let $\mathbb{G}(\Omega)$ be the set of all D-continuous interval functions. Assume that the set $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is closed under inclusion in the sense that*

$$\left.\begin{array}{c} f \in \mathcal{P}, \ \ g \in \mathbb{G}(\Omega) \\ g(x) \subseteq f(x), \ x \in \Omega. \end{array}\right\} \Longrightarrow g \in \mathcal{P}.$$

If $\mathcal{P} \subseteq \mathbb{G}(\Omega)$ is a linear space, then $\mathcal{P} \subseteq \mathbb{H}(\Omega)$.

Hence the H-operations "$\oplus$", "$\otimes$" cannot be extended further than $\mathbb{H}(\Omega)$ in a way preserving the algebraic structure of $C(\Omega)$.

3 Hausdorff approximations using step functions

3.1 Hausdorff distance and modulus of H-continuity

Let us recall that the Hausdorff distance (H-distance) $\rho(f, g)$ between two functions $f, g \in \mathbb{A}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$, is defined as the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$ [17], [21]. More precisely,

$$\rho(f, g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B||\}, \tag{2}$$

wherein $||.||$ is a norm in $\mathbb{R}^{n+1}$. In technical proofs presented in the sequel we assume that the norm in $\mathbb{R}^{n+1}$ is the maximum norm, that is for $A = (a_1, ..., a_{n+1})$ we have $||A|| = \max\{|a_1|, ..., |a_{n+1}|\}$. However, all statements remain true for any norm due to the equivalence of the norms in $\mathbb{R}^{n+1}$.

In the space of S-continuous functions on Ω, the H-distance satisfies the axioms of a metric. There is a natural connection between the H-continuous functions and the H-distance. For example, one can easily see that an S-continuous function f is H-continuous if and only if $\rho(\underline{f}, \overline{f}) = 0$. Indeed, it follows from the definition that f is H-continuous if and only if $F(\underline{f}) = F(\overline{f})$ or, equivalently, $\rho(F(\underline{f}), F(\overline{f})) = 0$. The link between the two concepts is further discussed below in terms of the modulus of H-continuity.

For $\delta > 0$, the operators I_δ and S_δ are defined for $f \in \mathbb{A}(\Omega)$ as follows

$$I_\delta(f)(x) = \inf\{\underline{f}(y) : y \in B_\delta(x)\}, \ x \in \Omega, \tag{3}$$

$$S_\delta(f)(x) = \sup\{\overline{f}(y) : y \in B_\delta(x)\}, \ x \in \Omega. \tag{4}$$

It is easy to see that in terms of Definition 2 we have

$$I(f)(x) = \sup_{\delta > 0} I_\delta(f)(x), \quad S(f)(x) = \inf_{\delta > 0} S_\delta(f)(x), \ x \in \Omega.$$

Definition 8. *The modulus of H-continuity $\tau(f;\delta)$ for given $f \in \mathbb{A}(\Omega)$ and $\delta > 0$ is the H-distance between $I_\delta(f)$ and $S_\delta(f)$, that is*

$$\tau(f;\delta) = \rho(I_\delta(f), S_\delta(f)) = \rho(F(I_\delta(f)), F(S_\delta(f))).$$

Theorem 5. *[21] An S-continuous function f is H-continuous iff $\lim_{\delta\to 0} \tau(f;\delta) = 0$.*

3.2 Interval step functions as an approximation tool

The usual concept of step-functions of a real argument can be extended to $\mathbb{H}(\Omega)$ as follows.

Definition 9. *A function $f \in \mathbb{H}(\Omega)$ is called a step function if there exists a collection $\{U_1, U_2, ..., U_m\}$ of open subsets of Ω with the following properties*

(i) $U_i \cap U_j = \emptyset$ for $i \neq j$,

(ii) the set $V = \bigcup_{i=1}^{k} U_i$ is dense in Ω,

(iii) for every $i \in \{1, 2, ..., k\}$, f is a real constant on U_i.

It is easy to see that a step function is completely determined by its values on the open set V. In fact we have $f = F(V, \Omega, f|_V)$. Similarly to the real step functions, an interval step function f assumes finite number of values, namely the constant values on the sets U_i, $i = 1, 2, ..., k$, and some real intervals with endpoints equal to the constant functional values. Further, we note that the set of step functions is a linear subspace of $\mathbb{H}(\Omega)$. Indeed, the sum of step functions is a step function and so is the product of a step function and a real number. In the next theorem we establish some approximation properties of the step functions.

Theorem 6. *Let $f \in \mathbb{H}(\Omega)$. For every $\varepsilon > 0$ there exists a step function φ such that $\rho(f, \varphi) < \varepsilon$.*

Proof. In view of Theorem 5, there exists $\delta > 0$ such that $\tau(f;\delta) < \varepsilon$. Then consider any collection of open sets $\{U_1, U_2, ..., U_m\}$ with the properties (i) and (ii) as given in Definition 9 and such that the diameter of each set is smaller than δ. These can be constructed for example by partitioning Ω via the planes $x_l = i\delta$, $i \in \mathbb{Z}$, $l = 1, 2, ..., n$. Since f assumes interval values only on a meagre subset of Ω, for every $i \in \{1, 2, ..., k\}$ there exists $x^{(i)} \in U_i$ such that $f(x^{(i)}) \in \mathbb{R}$. Define $\psi(x) = f(x^{(i)})$ for $x \in U_i$, $i = 1, 2, ..., k$, and $\varphi = F(V, \Omega, \psi)$.

We show that φ is the required function. First let us note that φ is a step function. Indeed, since ψ is continuous on any U_i, the operator F does not change the values of ψ on each of these sets, so that $\varphi(x) = \psi(x)$, $x \in V$. This implies both that φ is H-continuous and that it is a step function. Further, from the definition of φ it follows that

$$I_\delta(f) \leq \varphi \leq S_\delta(f).$$

Therefore, we have

$$\rho(f, \varphi) \leq \rho(I_\delta(f), S_\delta(f)) = \tau(f;\delta) < \varepsilon,$$

which proves the theorem.

In the special case of a real argument, the interval step-functions have a simple representation in terms of the Heaviside step function h given in (1). Indeed, when $\Omega = \mathbb{R}$, the sets U_i, $i = 1, ..., k$, associated with an interval step function f in terms of Definition 9 are open intervals of the form (d_{i-1}, d_i), where $d_0 = -\infty$, $d_k = +\infty$, and $d_1, d_2, ..., d_{k-1}$ is a finite increasing sequence of reals. Let $f(x) = c_i$ for $x \in (d_{i-1}, d_i)$. A familiar *rectangular pulse* on the interval $[d_{i-1}, d_i]$, $i = 1, ..., k-1$, is represented as

$$h(x - d_{i-1}) - h(x - d_i)$$

Then the step function f is given by

$$\begin{aligned} f(x) = c_1(1 - h(x - d_1)) \oplus c_2(h(x - d_1) - h(x - d_2)) \oplus ... \oplus c_{k-1}(h(x - d_{k-2}) - h(x - d_{k-1})) \\ \oplus\ c_k h(x - d_{k-1}) \ = \ c_1 + \sum_{i=1}^{k-1}(c_{i+1} - c_i)h(x - d_i). \end{aligned}$$

Note that f is discontinuous only at the points $d_1, ..., d_{k-1}$ where it assumes interval values. More precisely, we have

$$f(d_i) = \begin{cases} [c_i, c_{i+1}] \text{ if } c_i < c_{i+1} \\ [c_{i+1}, c_i] \text{ if } c_i > c_{i+1} \end{cases}.$$

For other approximation properties using Hausdorff metric adapted to resolution analysis one may consult Section 6 of [3].

4 Approximation by sigmoid functions

Sigmoid functions find multiple applications to neural networks and cell growth population models [14], [20]. A sigmoid function on $\mathbb{R}$ with a range $[a, b]$ is defined as a monotone function $s(t) : \mathbb{R} \to [a, b]$ such that $\lim_{t\to-\infty} s(t) = a$, and $\lim_{t\to\infty} s(t) = b$. One usually considers continuous (or even smooth) sigmoid functions. Within the class of H-continuous interval functions, the Heaviside step function is a particular case of sigmoid function.

4.1 Approximation by sigmoid logistic functions

An important class of smooth sigmoid functions arises from population growth models. A classical example is the familiar Verhulst population growth model, also known as *logistic model*. One can arrive to this model starting from the reaction equation $U + X \xrightarrow{k} X + X$, where U is a nutrient substance, X is a particular population and k is the specific growth rate of the population. The biological interpretation ot this reaction equation is that the nutrient U is utilized by the population X leading to the reproduction of the population. Denoting the biomass of X by x and the mass (concentration) of U by u and applying the mass action law, one obtains the dynamical system

$$\begin{aligned} du/dt &= -kxu, \\ dx/dt &= kxu, \\ u(0) &= u_0,\ x(0) = x_0. \end{aligned}$$

Noticing that $u' + x' = 0$, hence $u + x = x_0 + u_0 = \text{const} = a$, we can substitute $u = a - x$ in the differential equation for x to obtain the Verhulst differential equation $x' = kux = kx(a-x)$. The latter is usually written with a normalized rate constant $k := k/a$ as

$$\frac{dx}{dt} = \frac{k}{a}x(a-x) = kx\left(1 - \frac{x}{a}\right), \; x(0) = x_0. \tag{5}$$

The solution x to equation (5) passing through the point $(0, x(0) = x_0 = a/2)$ is the (basic) logistic sigmoid function:

$$s_0(t) = \frac{a}{1 + be^{-kt}}; \;\; b = \frac{a - x_0}{x_0} = 1. \tag{6}$$

In what follows we shall estimate the H-distance between a step function f and a logistic sigmoid function g. Without loss of generality we can consider the Heaviside step function $f = ah$ and the logistic sigmoid function (6): $g = s_0$. According to (2) the H-distance $\rho(f,g)$ between two functions $f, g \in \mathbb{A}(\Omega)$ for $\Omega \subset \mathbb{R}$ makes use of the maximum norm in R^2 so that the distance between the points $A = (t_A, x_A)$, $B = (t_B, x_B)$ in R^2 is $||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)$. In that case the H-distance $d = \rho(h, s_0)$ between the Heaviside step function ah and the sigmoid function (6) satisfies the relations $0 < d < a/2$ and $a - s_0(d) = d$, that is

$$(a - d)/d = e^{kd}, \;\; (0 < d < a/2). \tag{7}$$

Obviously $d \to 0$ implies $k \to \infty$ (and vice versa). From (7), a straightforward expression for the rate parameter k in terms of d follows:

$$k = \frac{1}{d} \ln \frac{a - d}{d} = O(d^{-1} \ln(d^{-1})). \tag{8}$$

4.2 Estimate for the H-distance in terms of the rate parameter

Relation (8) gives an estimate of the rate k in terms of the H-distance d. The following theorem gives a relation for the H-distance d in terms of the rate parameter k. For simplicity we assume $a = 1$, denoting thus in the sequel $s_0(t) = (1 + e^{-kt})^{-1}$.

Theorem 7. *The Hausdorff distance $d = \rho(h, s)$ between the Heaviside step function h_0 and the sigmoid Verhulst function s_0 can be expressed in terms of the reaction rate k as follows:*

$$d = \frac{\ln(k+1)}{k+1}\left(1 + O\left(\frac{\ln \ln(k+1)}{\ln(k+1)}\right)\right). \tag{9}$$

Proof. Assuming $a = 1$ relation (7) becomes $(1 - d)/d = e^{kd}$ for $0 < d < 1/2$. This implies $kd = \ln(1/d) + \ln(1 - d)$. In order to express d in terms of k, let us examine the function

$$f(d) = kd - \ln(1/d) - \ln(1 - d), \; 0 < d < 1/2.$$

From $\lim f(d)_{d \to 0, d>0} = -\infty$, $\lim f(d)_{d \to 1/2, d<1/2} = k/2 > 0$ we conclude that $f(d) = 0$ possesses a solution in $(0, 1/2)$. From $f'(d) = k + 1/d + 1/(1 - d) > 0$

we conclude that function f is strictly monotonically increasing, hence $f(d) = 0$ has an unique solution $d(k)$ in $(0, 1/2)$. For $k \to \infty$ we have $d(k) \to 0$, hence $\ln(1-d(k)) = -d(k)+O(d(k)^2)$. Consider then the function $g(d) = (k+1)d-\ln(1/d)$ which approximates function f with $d \to 0$ as $O(d^2)$; in addition $g'(d) > 0$. So we can further denote by $d(k)$ the (unique) zero of g and study g instead of f. We look for two reals d_- and d_+ such that $g(d_-) < 0$ and $g(d_+) > 0$ (leading to $g(d_-) < g(d(k)) < g(d_+)$ and thus $d_- < d(k) < d_+$). Trying $d_- = 1/(k + 1)$ and $d_+ = \ln(k + 1)/(k + 1)$ we obtain $g(1/(k + 1)) = 1 - \ln(k + 1) < 0$ and $g(\ln(k + 1)/(k + 1)) = \ln \ln(k + 1) > 0$ proving the estimates $1/(k + 1) < d(k) < \ln(k + 1)/(k + 1)$. To find a better lower bound we compute

$$g\left(\frac{\ln(k+1)}{k+1}\left(1-\frac{\ln\ln(k+1)}{\ln(k+1)}\right)\right) = \ln\left(1-\frac{\ln\ln(k+1)}{\ln(k+1)}\right) < 0.$$

We thus obtain

$$\frac{\ln(k+1)}{k+1} - \frac{\ln\ln(k+1)}{k+1} < d(k) < \frac{\ln(k+1)}{k+1},$$

which implies (9).

Remark. In the general case $a \neq 1$ one should substitute in (9) $k + 1$ by $k + a^{-1}$.

5 Conclusions

We briefly summarized some basic results (Theorems 1–5) about H-continuous functions and their application to problems in abstract areas such as real analysis, approximation theory, set-valued analysis and analysis of PDEs. We then formulated and proved a new result (Theorem 6) on the Hausdorff approximation of H-continuous functions by interval step functions defined on an open subset of $\mathbb{R}^n$. Finally we discussed some applications of H-continuous functions to biological dynamic processes, in particular, we considered the remarkable phenomenon that certain enzyme kinetic and population growth processes develop almost step-wise [16], [20]. Such processes are usually described or approximated by smooth sigmoid functions (especially in the theory of artificial neural networks). However, it is possible that H-continuous step-wise functions can be also conveniently used. To substitute a sigmoid function by a step function, we need to know the approximation error (Theorem 7). Biological processes are often very sensitive and can be effectively studied within the framework of interval analysis [19]. In addition, the input data coming from biological experiments are usually rather uncertain and thus can be represented as interval data. If these interval data are guaranteed (that is they include the measurement errors), then numerical methods and programming tools with automatic result verification can be used [18].

The presented results suggest that interval analysis, apart from being currently associated with numerical analysis, can also be considered as belonging to the field of real analysis. We may thus consider interval analysis as a bridge between real and numerical analysis, a bridge that extends both subjects and unifies them into a common scientific area.

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